REPRESENTATION AND ESTIMATION TECHNIQUES FOR CYCLOSTATIONARY RANDOM PROCESSES

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13. ABSTRACT

Many communication and control systems employ signal formats that involve some form of periodic processing operation. Familiar examples are signals produced by samplers, scanners, multiplexers, or modulators. Very often these signals can be modelled as cyclostationary processes, i.e., processes whose statistical properties, such as mean and autocorrelation, fluctuate periodically with time. Filters designed to extract signals of this type from a noise background can exhibit dramatically improved performance when the periodic nature of the statistics are taken into account, rather than using the more conventional "time-average" statistical approach. Some techniques for solving for the optimum filter and a video signal example are discussed.

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I. Introduction:

A random signal process which is produced by a periodic sampling, scanning or synchronous multiplexing operation can often be modelled by a process which exhibits periodicity in its mean and autocorrelation functions. A process, $x(t)$, is said to be cyclostationary in the wide sense (with period T) if

$$E[x(t)] = E[x(t + T)] \qquad (1)$$

$$k_{xx}(s, t) = E[x(s)x^*(t)] = k_{xx}(s + T, t + T) \qquad (2)$$

for all s and t. Bennett [1] introduced the term, "cyclostationary", to denote this class of processes in his treatment of synchronously timed pulse sequences used in digital data transmission. Other investigators [2]-[5] have used terms such as "periodically stationary", "periodically correlated", and "periodic nonstationary" to denote this same class.

One important example of a cyclostationary process is the synchronous pulse amplitude modulation (PAM) signal. Assuming that the pulse amplitudes $\{a_k\}$ form a stationary sequence of random variables, we have

$$x(t) = \sum_{k=-\infty}^{\infty} a_k s(t - kT) \qquad (3)$$

where
$$E[a_k] = \bar{a}$$
$$E[a_{k+m} a_k] = \alpha_m$$
for all k

In this case the mean and autocorrelation are given by [6]

$$E[x(t)] = \bar{a} \sum_{k=-\infty}^{\infty} s(t - kT)$$

$$k_{xx}(t + \tau, t) = \sum_{m=-\infty}^{\infty} \alpha_m q(t, \tau + mT)$$

where

$$q(t, \tau) \stackrel{\Delta}{=} \sum_{k=-\infty}^{\infty} s(t + \tau - kT)s(t - kT) \qquad (4)$$

By inspection, the expressions in (4) are periodic T in t and hence the PAM signal is a cyclostationary process. Typically, the pulse amplitudes could be the result of uniformly sampling a wide-sense stationary process. For example, the output of a conventional sample-and-hold device can be represented by (3) where s(t) is a unit amplitude rectangular pulse of width T. Other forms of pulse modulation such as frequency-shift keying (FSK) and phase-shift keying (PSK) will also yield cyclostationary processes [7]. The conventional formats for time-division and frequency-division multiplexing of signals, and video signals generated by rectangular scanning of a two-dimensional field [9] provide additional examples of cyclostationary processes.

In the communication system examples cited above, it is clear that the receiver used for demultiplexing or reconstruction of the signal into its original format will require accurate timing information for satisfactory operation. This information is normally supplied by inserting synchronizing or framing pulses or by superimposing some other form of periodic signal on the random signal process. Another function of the receiver is to remove, to the extent possible, the effects of noise interference. It is

of interest to determine the amount of improvement in filtering that results from the cyclostationary character of the received signal process and employing filters with time-variable elements whose periodic fluctuations are "locked in" to the fundamental period of the cyclostationary process. We have obtained solutions for optimum periodically-varying filters and expressions for improvement in performance over that obtainable by the best time-invariant filters. The receiver structures implied by these solutions have a form dictated by the choice of representation used for the cyclostationary process. Two representation techniques which have been found generally useful are briefly presented in the following section.

II. Representation and Properties of Cyclostationary Processes:

We say that a process has a harmonic representation when expressed in the form

$$x(t) = \sum_{n=-\infty}^{\infty} a_n(t) \exp [j2\frac{\pi nt}{T}] \tag{5}$$

where choosing

$$a_n(t) = \int_{-\infty}^{\infty} v(t - \tau) \exp [-j\frac{2\pi n\tau}{T}] x(\tau) d\tau$$

$$v(t) = \frac{1}{\pi t} \sin \frac{\pi t}{T} \tag{6}$$

makes the mean-squared value of the difference of both sides of (5) vanish. Generation of the sequence of bandlimited random processes, $\{a_n(t)\}$, is illustrated in Fig. 1. If $x(t)$ is cyclostationary, then we can show that $\{a_n(t)\}$ is jointly wide-sense stationary[1] and we define:

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[1]Ogura [5] has presented a discussion of harmonic representation; however, he observes only that the individual terms in (5) are wide sense stationary. They are not jointly wide sense stationary, as the coefficient processes are, and this point has important consequences in the estimation or filtering problem.

$$r_{nm}(s - t) = E[a_n(s)a_m^*(t)] \qquad (7)$$

with the result that

$$k_{xx}(s, t) = \sum_n \sum_m r_{nm}(s - t) \exp [j \frac{2\pi}{T} (ns - mt)] \qquad (8)$$

There are certain properties of the infinite-dimensional matrix, $\underline{r}(\cdot)$, or its Fourier transform $\underline{R}(\cdot)$ which will be useful in the estimation problem.

1) $\underline{R}(f)$ is bandlimited to the frequency interval, $|f| \leq \frac{1}{2T}$. This is because the $\{a_n(t)\}$ are similarly bandlimited.

2) $x(t)$ is a wide-sense stationary process if, and only if, $\underline{r}(\cdot)$ is a diagonal matrix. This can be seen by direct substitution into (8).

3) Let $x(t)$ be the result of a time-invariant filtering operation on a cyclostationary $x'(t)$, with representation $\underline{R}'(\cdot)$, given by

$$x(t) = \int_{-\infty}^{\infty} h(t - \tau) x'(\tau)d\tau$$

then we have

$$R_{nm}(f) = V(f) H(f + \frac{n}{T}) H^*(f + \frac{m}{T}) R'_{nm}(f) \qquad (9)$$

where $V(f)$ is a rectangular function of width, $1/T$, which is the Fourier transform of $v(t)$ in (6). Using this result with $H(\cdot)$ corresponding to an ideal lowpass filter with cutoff at W, we see that a bandlimited process, $x(t)$, can be represented by a finite matrix of order M where $M \leq 2$ TW.

4) Let $x(t)$ be derived from a cyclostationary $x'(t)$ where an uncertainty

in the time origin or "phase" of the process has been introduced, i.e., let $x(t) = x'(t + \delta)$ where δ is a random variable with probability density function, $p_\delta(\cdot)$. The modification in the representation is expressed by

$$R_{nm}(f) = P_\delta(\frac{m-n}{T}) \ R'_{nm}(f) \tag{10}$$

where $P_\delta(\cdot)$ is the Fourier transform of $p_\delta(\cdot)$ so we know that

$$P_\delta(\frac{n}{T}) \le P_\delta(0) = 1 \tag{11}$$

In particular, if δ is uniformly distributed over the interval $[0, T]$, then $P(\frac{n}{T}) = 0$ for $n \ne 0$ and $\underline{R}(f)$ is diagonal. Thus if a cyclostationary process is completely "phase randomized" it becomes a stationary process. Its autocorrelation function would be

$$k_{xx}(\tau) = \sum_n r'_{nn}(\tau) \ \exp \ [j \ 2\pi \ \frac{n\tau}{T}]$$

$$= \frac{1}{T} \int_0^T k_{x'x'}(t + \tau, t)dt \tag{12}$$

which is simply a time-averaged version of the nonstationary correlation.

An alternative form of representation which affords a simpler solution to the filtering problem in many cases of practical interest is the time-series representation. This representation has the form

$$x(t) = \sum_{i=1}^M \sum_{n=-\infty}^\infty \alpha_{ni} \ \phi_i(t - nT) \tag{13}$$

where the $\{\phi_i(t)\}$ are "doubly-orthogonal" in the sense that

$$\int_{-\infty}^\infty \phi_i(t - nT)\phi_j^*(t - mT)dt = \delta_{ij}\delta_{nm} \tag{14}$$

and the process is represented by the random variables

$$\alpha_{ni} = \int_{-\infty}^{\infty} x(t) \, \phi_i^*(t - nT)dt \qquad (15)$$

This leads to a representation for the autocorrelation given by

$$k_{xx}(s, t) = \sum_n \sum_m \sum_i \sum_j A_{n-m}^{ij} \, \phi_i(s - nT) \, \phi_j^*(t - mT) \qquad (16)$$

where

$$A_{n-m}^{ij} = E[\alpha_{ni}\alpha_{mj}^*]$$

$$\int_{-\infty}^{\infty} \int k_{xx}(t + nT - mT, \tau) \, \phi_i^*(t) \, \phi_j(\tau)dtd\tau \qquad (17)$$

The time-series representation is particularly appropriate for the various forms of digital pulse modulation. For example, in the simplest case, PAM with $\{s(t - nT)\}$ forming an orthonormal sequence, we have $M = 1$ and

$$\phi_i(t) = s(t)$$

$$\alpha_{ni} = a_n \qquad (18)$$

and the A_{n-m} matrices in (17) are simply the scalar correlation values of the pulse amplitudes as given in (3).

A more general application of the time-series representation for an arbitrary cyclostationary process results from a Karhunen-Loeve expansion of the process over each T second interval [6], using the same basis functions for each interval. This has proved particularly effective for the random video process described later.

III. Continuous Waveform Estimation:

One of the estimation problems of interest is the problem of filtering to reproduce a faithful replica of the signal waveform, at each point in time, in the situation where the received waveform has an additive noise component. In the following, we consider the case where there is no channel dispersion and the noise is white, although more general results with these restrictions removed, follow directly.

Accordingly, the estimation problem we consider here is evaluation of the performance of the noncausal, periodically-variable filter which gives least mean-squared error reconstruction of a cyclostationary signal process with additive white noise having spectral density, N_o. The impulse response function, $h(\cdot, \cdot)$, for the filter which minimizes

$$I(t) = E[\{x(t) - \int_{-\infty}^{\infty} h(t, s)z(s)ds\}^2] \qquad (19)$$

where $z(t)$ is signal plus white noise, must satisfy the orthogonality condition [6]

$$\int_{-\infty}^{\infty} k_{xx}(s, \sigma)h(t, \sigma)d\sigma + N_o h(t, s) = k_{xx}(s, t) \qquad \text{for all } s \text{ and } t \qquad (20)$$

Substitution of (20) into (19) and rearranging terms yields a compact expression for the minimum mean-squared error.

$$I_{min}(t) = N_o h(t, t) \qquad (21)$$

The mean-squared error in (21) is periodic T in t. As a performance functional, we select the value of $I_{min}(t)$ averaged over one period, i.e.,

$$J_0 = \frac{N_0}{T} \int_0^T h(t, t)dt \qquad (22)$$

Using the harmonic representation scheme of (5), the optimum filter can be interpreted as an estimator of the jointly wide-sense stationary sequence, $(a_n(t))$. This is accomplished by forming the corresponding sequence of processes for the received signal by means of the structure shown in Fig. 1. The estimate of the nth process is a linear combination of time-invariant operations on the received signal processes which is then multiplied by $\exp [j \, 2\pi \frac{nt}{T}]$ to form one component of the filter output. The structure of this estimator is shown in Fig. 2 and the overall filter is obtained as a cascade of the structures in Figs. 1 and 2. The filter is realized as a bank of input and output modulators interconnected by a matrix of time-invariant, bandlimited $\frac{1}{2T}$, filter paths. We can then write the impulse response as

$$h(t, s) = \sum_n \sum_k g_{nk}(t - s) \exp [j \, \frac{2\pi}{T} (nt - ks)] \qquad (23)$$

Now using (23) and (8) in (20) and performing the integration, the coefficients of the exponentials in t and s are equated and the orthogonality condition is satisfied by a $\underline{g}(\cdot)$ matrix whose Fourier transform is the solution of

$$\underline{G}(f) \, [\underline{R}^a(f) + N_0 \underline{I}] = \underline{R}^a(f); \quad |f| \leq \frac{1}{2T} \qquad (24)$$

where $\underline{R}^a(\cdot)$ denotes the conjugate transpose of $\underline{R}(\cdot)$.

Using (23) in (22), the expression for optimum performance becomes

$$J_0 = N_0 \sum_n g_{nn}(0) = N_0 \int_{-\infty}^{\infty} tr \ \underline{G}(f)df \qquad (25)$$

To evaluate the performance of the best time-invariant filter for a cyclostationary signal relative to that of the best periodically-variable filter, we use $h(t - s)$ in (19) and find the condition which minimizes the time-averaged value of $I(t)$. The constraint of time-invariance is equivalently imposed by requiring that tne $\underline{g}(\cdot)$ matrix for (23) be diagonal. The new orthogonality condition has the simple solution,

$$G_{nn}(f) = \frac{R_{nn}(f)}{R_{nn}(f) + N_0} . \qquad (26)$$

and since $\underline{G}(\cdot)$ is diagonal, the filter transfer function is

$$H(f) = \sum_n G_{nn}(f - \frac{n}{T}) \qquad (27)$$

Taking into account the bandlimited nature of $R_{nn}(f)$ in (26), the sum in (27) can also be written as

$$H(f) = \frac{\sum_n R_{nn}(f - \frac{n}{T})}{\sum_n R_{nn}(f - \frac{n}{T}) + N_0} = \frac{K(f)}{K(f) + N_0} \qquad (28)$$

The expression (28) is the familiar solution for the noncausal Wiener filter for a stationary signal with power spectral density $K(f)$ in white noise [6]. Now referring to (12), we see that $K(f)$ is actually the power spectral density for the (stationary) phase-randomized version of the cyclostationary process. Thus we make the interesting observation that

the best time-invariant filter for the cyclostationary process is identical
to the best filter for the phase-randomized version of the process. Further-
more, this filter is the result of simply disregarding the crosscorrelation
among the $\{a_n(t)\}$ processes in making the estimation. The performance
functional for the time-invariant filter is

$$J_1 = N_0 h(0) = N_0 \sum_n \int_{-\infty}^{\infty} \frac{R_{nn}(f - \frac{n}{T}) \, df}{R_{nn}(f - \frac{n}{T}) + N_0} \tag{29}$$

When the time-series representation approach (13) is used, the optimum
filter exhibits a similar representation.

$$h(t, s) = \sum_{\ell} \sum_{k} \sum_{p} \sum_{q} H_{\ell-k}^{qp} \, \phi_q(s - \ell T) \, \phi_p^*(t - kT) \tag{30}$$

Substituting (30) and (16) into the orthogonality condition (20) and
performing the indicated integration by making use of (14), we obtain an
equation whose terms have the same form as (16) and (30). Equating coeffi-
cients in this equation, the orthogonality condition is satisfied by the
solution of

$$\sum_m \underline{A}_{n-m} \, \underline{H}_m + N_0 \, \underline{H}_n = \underline{A}_n; \quad \text{for all } n \tag{31}$$

where the matrices $\underline{A}_n$ and $\underline{H}_n$ in (31) have elements as indicated by the
superscripts in (16) and (30).

The first term in (31) is a discrete convolution, so z-transform
techniques can be used to express the solution. To this end we define

$$\underline{B}(f) = \sum_n \underline{A}_n \, \exp[j2\pi nTf]$$

$$\underline{L}(f) = \sum_n \underline{H}_n \, \exp[j2\pi nTf] \tag{32}$$

and (31) is expressed as

$$\underline{B}(f)\underline{L}(f) + N_0\underline{L}(f) = \underline{B}(f) \tag{33}$$

The optimum filter is given by

$$\underline{H}_n = T \int_{-1/2T}^{1/2T} \underline{L}(f) \exp[-j2\pi nTf]df \tag{34}$$

where $\underline{L}(f) = [\underline{B}(f) + N_0\underline{I}]^{-1}\underline{B}(f)$.

The solution for the optimum filter (30) indicates the structure of filter as outlined in Figure 3. The structure involves a bank of time-invariant filters whose outputs are sampled every T seconds. The appropriate linear combinations of these samples are formed by the time-invariant, multiport, sampled-data filter, characterized by the $\underline{H}_n$ matrices, and the output signal is reconstructed by impulsing a similar bank of output filters. The performance functional for this filter is given by

$$J_0 = \frac{N_0}{T} \, \text{tr} \, \underline{H}_0 = N_0 \sum_i \int_{-1/2T}^{1/2T} L^{ii}(f)df \tag{35}$$

Solutions for specific examples related to time-division and frequency-division multiplexing of an arbitrary number of independent, band-limited signal processes have been presented [8]. In the following section, we illustrate the application of the time-series method of representation for finding the optimum filter for the random video process.

IV. Video Signal Process:

We assume that the video signal results from scanning a two-dimensional visual pattern using the conventional rectangular raster (without interlace). The visual pattern is modeled with a two-dimensional random step function giving a stationary autocorrelation with exponential form which is separable in the horizontal and vertical directions [9]. Neglecting frame-to-frame correlation, the scanner output is a cyclostationary process with period T equal to the line scan interval. Consider any two time instants t_1 and t_2 where t_2 occurs in the mth line after the one which contains t_1. Then the normalized autocorrelation of the scanner output is given by

$$k_{xx}(t_1, t_2) = \rho^m \exp[-2\pi f_0 |t_1 - t_2 + mT|] \tag{36}$$

where the parameter f_0 characterizes correlation in the horizontal direction and ρ is the line-to-line correlation.

For the time-series representation (13), we choose the $\phi_i(t)$ as the normalized solutions of

$$\int_0^T \exp[-2\pi f_0 |t - s|]\phi_i(s)ds = \lambda_i \phi_i(t) \qquad \text{for } 0 \le t \le T \tag{37}$$

and we are, in effect, making a separate Karhunen-Loeve expansion over each line-scan interval. The eigenfunctions in (37) are cosine and sine functions [6] whose frequencies, respectively, are given by the solutions of

$$\tan \pi T f_i = f_0/f_i \quad \text{and} \quad \tan \pi T f_i = -f_i/f_0 \tag{38}$$

The eigenvalues are

$$\lambda_i = \frac{1}{\pi f_o} \left[1 + (f_i/f_o)^2\right]^{-1} \tag{39}$$

and the matrix sequence characterizing the autocorrelation (16), (17) has an especially convenient form.

$$\underline{A}_m = \rho^{|m|} \underline{\Lambda} \tag{40}$$

where $\underline{\Lambda}$ is a diagonal matrix whose elements are the λ_i. $\underline{B}(f)$ in (32) is obtained by summing the double-sided geometric series and we get

$$\underline{B}(f) = \frac{1 - \rho^2}{(1 - \rho)^2 + 4\rho \sin^2 \pi Tf} \underline{\Lambda} \tag{41}$$

Solving the orthogonality condition (33) for the filtering problem and evaluating the performance functional (35) gives the result.

$$J_o = \frac{N_o}{T} \sum_i \lambda_i \left| \left\{\lambda_i + N_o \left(\frac{1 + \rho}{1 - \rho}\right)\right\} \left\{\lambda_i + N_o \left(\frac{1 - \rho}{1 + \rho}\right)\right\} \right|^{-1/2} \tag{42}$$

The performance (42) has been numerically evaluated for the specific case of a 500-line, square format assuming that the visual pattern has the same correlation in horizontal and vertical directions. This requires $\rho = \exp[-2\pi f_o T/500]$. Assuming approximately equal resolution requirements in horizontal and vertical directions, the essential bandwidth of the signal is 500/2T so we take $\delta = T/500N_o$ as a measure of the signal-to-noise power ratio. Results for various values of ρ and δ are shown in Figure 4.

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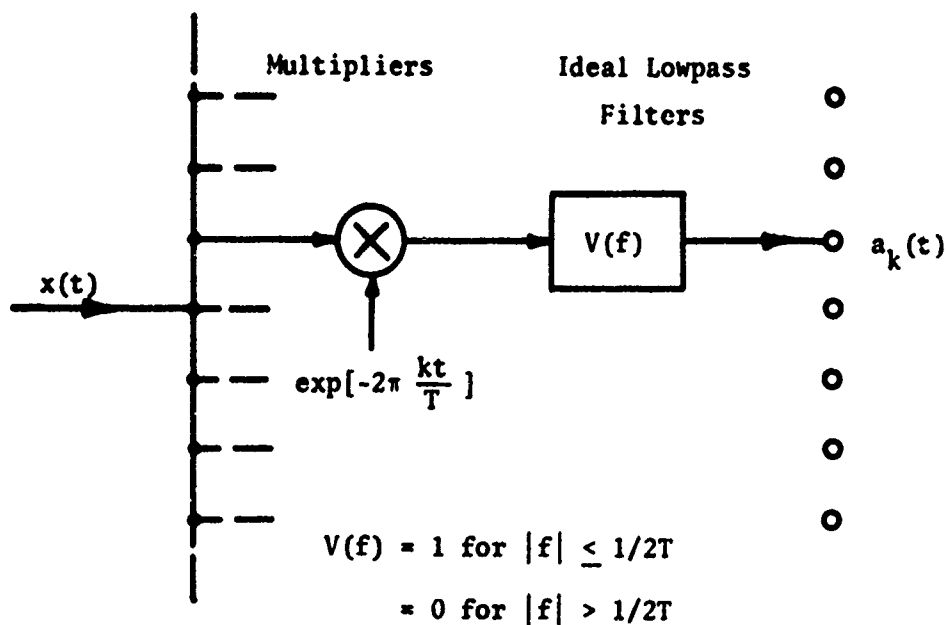
Fig. 1. Harmonic representation.

$$V(f) = 1 \text{ for } |f| \leq 1/2T$$
$$= 0 \text{ for } |f| > 1/2T$$
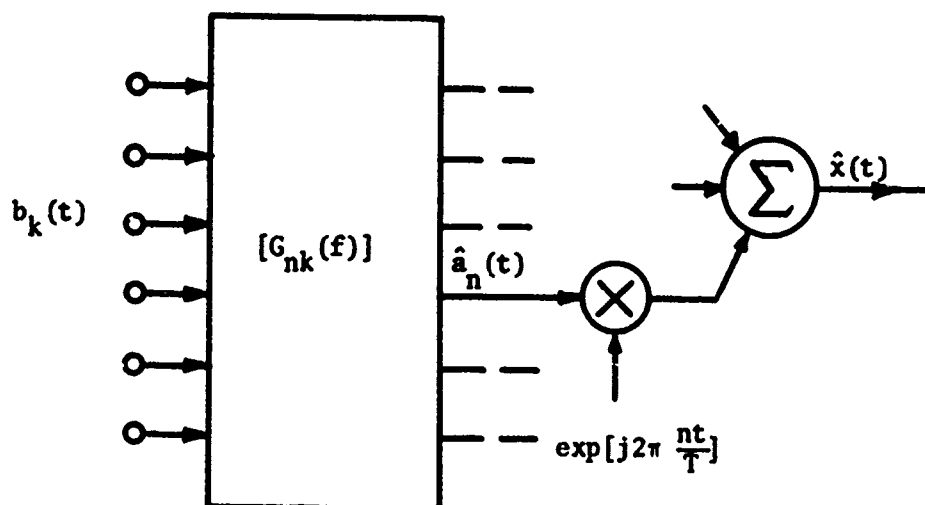
$$b_k(t) = \int_{-\infty}^{\infty} v(t - \tau)\exp[-j2\pi \frac{k}{T}]z(\tau)d\tau = a_k(t) + n_k(t)$$

Fig. 2. Estimation using harmonic representation
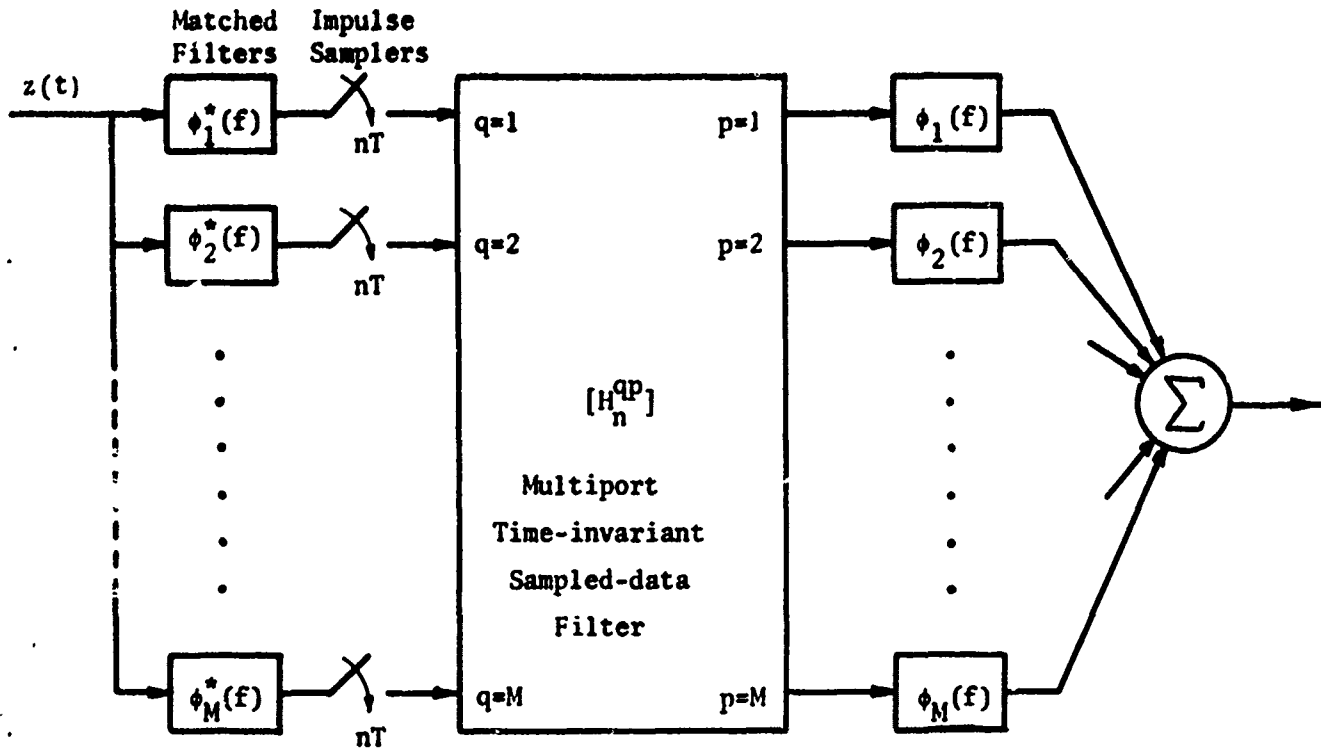of received signal.

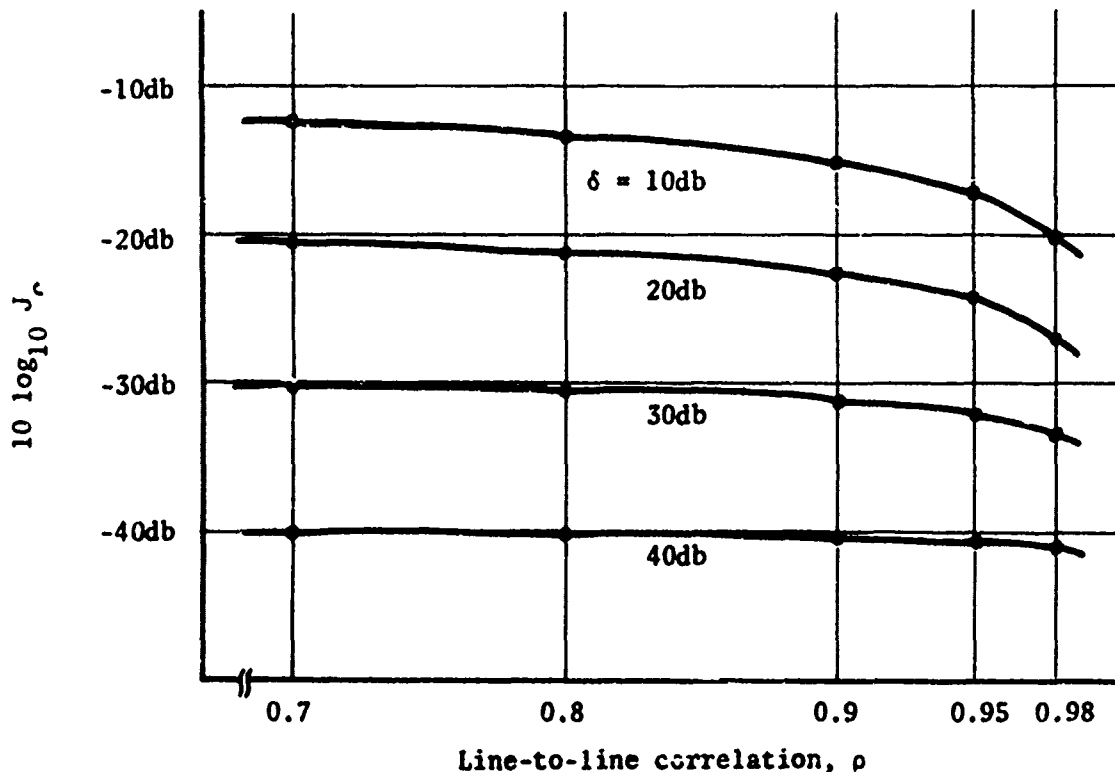Fig. 3. Structure of optimum filter from time-series approach.

Fig. 4. Calculated performance for the video signal filtering problem.